Tables of Values of the Modified Mathieu Functions

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1. Introduction. Mathieu functions are encountered in physical problems involving elliptical boundaries. When the method of separation of variables is applied to the wave equation or the diffusion equation expressed in elliptical coordinates, there result [1, p. 170] the Mathieu and modified Mathieu equations,

$$\frac{d^2y}{du^2} + (a - 2q\cos 2u)y = 0 \tag{1}$$

$$\frac{d^2y}{du^2} - (a - 2q\cosh 2u)y = 0. \tag{2}$$

The notation used is that of Goldstein, Ince and McLachlan. Solutions to the modified Mathieu equation (2) can be assumed to be of the form

$$y = Ce_{2n}(u, q) = \sum_{r=0}^{\infty} A_{2r}^{(2n)} \cosh 2ru \tag{3a}$$

$$y = Ce_{2n+1}(u, q) = \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)} \cosh (2r + 1)u \tag{3b}$$

$$y = Se_{2n+1}(u, q) = \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)} \sinh (2r + 1)u \tag{3c}$$

$$y = Se_{2n+2}(u, q) = \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)} \sinh (2r + 2)u. \tag{3d}$$

When the above infinite hyperbolic series are substituted into equation (2), recurrence relationships may be derived for allowable values of the characteristic numbers a_{2n+p} ($p = 0$ or 1) for the even functions, or b_{2n+s} ($s = 1$ or 2) for the odd functions for a given value of q. Recurrence equations also give the allowable values of the Fourier coefficients $A_{2r+p}^{(2n+p)}$ and $B_{2r+s}^{(2n+s)}$

The formulas for both the characteristic numbers and the Fourier coefficients associated with each type of solution [1, p. 29, 37] are given below:

For $y = Ce_{2n}(u, q)$

$$a_{2n} = \frac{2q^2}{a - 2^2 -} \frac{q^2}{a - 4^2 -} \frac{q^2}{a - 6^2 -} \cdots \frac{q^2}{a - (2r)^2 -} \cdots \tag{4a}$$

$$v_{2r-2} = \frac{q}{a - (2r)^2 - qv_{2r}}, \qquad r \geqq 2 \tag{4b}$$

$$v_0 = \frac{2q}{a - (2r)^2 - qv_2}, \qquad r = 1 \tag{4c}$$

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where

$$v_{2r-2} = \frac{A_{2r}^{(2n)}}{A_{2r-2}^{(2n)}}. \tag{4d}$$

For $y = Ce_{2n+1}(u, q)$

$$a_{2n+1} = 1 + q + \frac{q^2}{a - 3^2 -} \frac{q^2}{a - 5^2 -} \cdots \frac{q^2}{a - (2r+1)^2 -} \cdots \tag{5a}$$

$$v_{2r-1} = \frac{q}{a - (2r+1)^2 - qv_{2r+1}}, \qquad r \geqq 2 \tag{5b}$$

$$v_1 = \frac{a - 1 - q}{q}, \qquad r = 1. \tag{5c}$$

For $y_{2n+1} = Se_{2n+1}(u, q)$

$$b_{2n+1} = 1 - q + \frac{q^2}{a - 3^2 -} \frac{q^2}{a - 5^2 -} \cdots \frac{q^2}{a - (2r+1)^2 -} \cdots \tag{6a}$$

$$v_{2r+1} = \frac{q}{a - (2r+1)^2 - qv_{2r+1}}, \qquad r \geqq 2 \tag{6b}$$

$$v_1 = \frac{a - 1 + q}{q}, \qquad r = 1. \tag{6c}$$

For $y_{2n+2} = Se_{2n+2}(u, q)$

$$b_{2n+2} = 4 + \frac{q^2}{a - 4^2 -} \frac{q^2}{a - 6^2 -} \cdots \frac{q^2}{a - (2r)^2 -} \cdots \tag{7a}$$

$$v_{2r-2} = \frac{q}{a - (2r)^2 - qv_{2r}}, \qquad r \geqq 3 \tag{7b}$$

$$v_2 = \frac{a - 4}{q}, \qquad r = 2. \tag{7c}$$

As a consequence [1, p. 24] of the Goldstein-Ince normalization

$$\int_0^{2\pi} y^2\, du = \pi, \text{ then for } y = Ce_{2n}(u, q), \qquad 1 = 2\,[A_0^{(2n)}]^2 + \sum_{r=1}^{\infty} [A_{2r}^{(2n)}]^2 \tag{8}$$

and for the remaining three types,

$$1 = \sum_{r=0}^{\infty} [A_{2r+1}^{(2n+1)}]^2 = \sum_{r=0}^{\infty} [B_{2r+1}^{(2n+1)}]^2 = \sum_{r=0}^{\infty} [B_{2r+2}^{(2n+2)}]^2. \tag{9}$$

By dividing equations (8) and (9) through by $[A_0^{(2n)}]^2$ there are obtained equations of the form

$$\left[\frac{1}{A_0^{(2n)}}\right]^2 = 2 + \left[\frac{A_2^{(2n)}}{A_0^{(2n)}}\right]^2 + \left[\frac{A_4^{(2n)}}{A_0^{(2n)}}\right]^2 + \left[\frac{A_6^{(2n)}}{A_0^{(2n)}}\right]^2 + \cdots$$
$$= 2 + v_0^2 + (v_2 \cdot v_0)^2 + (v_4 \cdot v_2 \cdot v_0)^2 + \cdots. \tag{10}$$

2. Computation of the Modified Mathieu Functions. For a given value of q, the characteristic number is found by using a trial, error and interpolation method

using the appropriate transcendental continued fraction of the form of equation (4a). The computation was generally started with the 21st term equal to zero, and then the 20th, 19th, etc., terms were computed in turn.

The v_{2r} are all known from equations of the type (4b) and (4c) above and therefore A_0 can be evaluated from equation (10). Equation (4d) will then give the value of the remaining A_{2r}. The value of the modified Mathieu function can then be computed by summing the appropriate hyperbolic series.

In [2], E. L. Ince has given tables for the characteristic numbers, Fourier coefficients and values of the Mathieu functions satisfying equation (1). The method used by Ince was to apply the above formulas; the same method was used by the author to evaluate the modified Mathieu functions satisfying equation (2). The numerical work was accomplished by using an IBM 650 Digital Computer and the Wolontis Interpretive System of coding. The characteristic numbers and the Fourier coefficients are the same for both the ordinary and the modified functions. However, the problem of summing the hyperbolic series is much more difficult than that of summing the trigonometric series. Whereas the cos $2ru$ term is bounded by plus or minus one, the sinh $2ru$ and cosh $2ru$ approach infinity as r tends to infinity. Therefore many more terms of the hyperbolic series must be considered before it is possible to generate a product $A_{2r}^{(2n)}$ cosh $2ru$ which will be negligibly small.

As more terms are taken, difficulties in computation are encountered. The v_{2r} can be made as small as desired by choosing r sufficiently large. In most cases after the characteristic number had been found by using the appropriate transcendental continued fraction, the computation of the Fourier coefficients was started by letting $v_{40} = 0$, and then compute the v_{38}, v_{36}, $\cdots$ v_0, in turn. As these values were computed, they were stored and used to form the products v_0^2, $(v_2 \cdot v_0)^2$, etc., for equation (10). It is to be noted that the smallest number which can be computed using the Wolontis Interpretive System is 10^{-50}. At first consideration this may not appear to be a serious limitation. However, in order to obtain, for example, fifteen coefficients up to $A_{28}^{(2n)}$, from the relations $A_{2r+2} = v_{2r}A_{2r}$, the v_{26} must be available. For it to be available the product of $(v_{26} \cdot v_{24} \cdot \cdots \cdot v_0)^2$ must be greater than 10^{-50}, or the unsquared value must be greater than 10^{-25}. Except for v_0, v_2 and v_4 which may sometimes be greater than one, all the remaining values are less than one. In fact, for convergence, $v_{2r} \to 0$ as $r \to \infty$. If, for example, the "average" size of the v_{2r} is of the order of 10^{-2}, only twelve v_{2r} can be multiplied together before their product becomes less than 10^{-25}. Therefore, for this example, the maximum number of coefficients that can be generated will be twelve or thirteen. A machine error stop was avoided by counting the products as they were formed and at the same time checking their magnitude. The maximum number of coefficients generated was twenty; that is, up to and including coefficient $A_{38}^{(2n)}$. The number of coefficients generated will depend on the values of q and order $2n$ being considered.

3. Description of Computer Programs. A computer program was developed for the IBM 650 Digital Computer which will compute the characteristic numbers and Fourier coefficients and will sum the appropriate series in order to find the value of either the ordinary or the modified Mathieu function. A complete description of this program titled "A Program for the Computation of Mathieu and Modified Mathieu Functions" is available from the author and will be published separately.

In order to ensure five figure accuracy for parameters in the ranges $0 \leqq q \leqq 20$, $0 \leqq n \leqq 3$ and $0 \leqq u \leqq 1.0$, the program can use a maximum of twenty terms. For parameters outside the given ranges, the accuracy is unknown. However, the user can estimate the accuracy by noting the magnitude of the last term added to the series.

4. Discussion of Tables. Tables 1–12 are provided herein for values of the four types of modified Mathieu function $Ce_{2n}(u, q)$, $Ce_{2n+1}(u, q)$, $Se_{2n+1}(u, q)$ and $Se_{2n+2}(u, q)$, for the ranges $q = 1(1)10(2)20$, $u = 0.1(0.1)1.0$ and $n = 0, 1, 2$. The Fourier coefficients computed in order to obtain these values were compared with existing tables [1] for over 150 values, and no discrepancy was found in the first seven digits of any coefficient generated. Also, the magnitude of the last term added in the series was noted to ensure the accuracy of the last position published.

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1. H. W. McLachlan, *Theory and Application of Mathieu Functions*, Oxford University Press, London, 1947.
2. E. L. Ince, "Tables of the elliptical cylinder functions," Roy. Soc., Edinburgh, *Proc.*, v. 52, 1932, p. 355–423.

TABLE 1

Values of Modified Mathieu Function $Ce_0(u, q)$

q	$u = 0.1$	$u = 0.2$	$u = 0.3$	$u = 0.4$	$u = 0.5$
1	.38010	.36588	.34209	.30863	.26554
2	.19704	.18047	.15340	.11679	.07224
3	.10989	.09505	.07142	.04076	.00576
4	.06564	.05329	.03412	.01034	−.01491
5	.04130	.03127	.01612	−.00179	−.01932
6	.02702	.01895	.00709	−.00623	−.01807
7	.01822	.01174	.00249	−.00734	−.01515
8	.01259	.00738	.00017	−.00705	−.01206
9	.00887	.00468	−.00094	−.00624	−.00931
10	.00636	.00298	−.00142	−.00529	−.00705
12	.00341	.00118	−.00153	−.00355	−.00388
14	.00190	.00042	−.00126	−.00229	−.00203
16	.00110	.00010	−.00095	−.00144	−.00101
18	.00065	−.00002	−.00069	−.00089	−.00045
20	.00040	−.00007	−.00049	−.00054	−.00016
q	$u = 0.6$	$u = 0.7$	$u = 0.8$	$u = 0.9$	$u = 1.0$
1	.21308	.15200	.08378	.01097	−.06241
2	.02222	−.02961	−.07835	−.11775	−.14074
3	−.02984	−.06129	−.08309	−.08980	−.07753

TABLE 1 (*Continued*)

4	−.03764	−.05332	−.05766	−.04792	−.02464
5	−.03275	−.03843	−.03382	−.01887	.00267
6	−.02529	−.02528	−.01712	−.00270	.01269
7	−.01841	−.01551	−.00677	.00478	.01379
8	−.01290	−.00880	−.00094	.00728	.01122
9	−.00876	−.00443	.00199	.00722	.00773
10	−.00576	−.00172	.00319	.00606	.00457
12	−.00219	.00073	.00314	.00319	.00054
14	−.00055	.00127	.00211	.00113	−.00092
16	.00011	.00112	.00116	.00007	−.00106
18	.00032	.00081	.00051	−.00033	−.00074
20	.00034	.00052	.00014	−.00039	−.00038

TABLE 2

Values of Modified Mathieu Function $Ce_1(u, q)$

q	$u = 0.1$	$u = 0.2$	$u = 0.3$	$u = 0.4$	$u = 0.5$
1	.85596	.85372	.84884	.83954	.82334
2	.67799	.66081	.63071	.58567	.52324
3	.50011	.47318	.42746	.36215	.27703
4	.35310	.32216	.27089	.20034	.11323
5	.24612	.21525	.16523	.09889	.02144
6	.17252	.14384	.09841	.04040	−.02332
7	.12249	.09685	.05717	.00848	−.04153
8	.08824	.06581	.03192	−.00795	−.04594
9	.06445	.04509	.01652	−.01561	−.04369
10	.04769	.03109	.00721	−.01840	−.03863
12	.02697	.01493	−.00151	−.01739	−.02697
14	.01582	.00713	−.00408	−.01366	−.01732
16	.00955	.00329	−.00433	−.00994	−.01054
18	.00591	.00138	−.00379	−.00696	−.00612
20	.00373	.00044	−.00306	−.00475	−.00337
q	$u = 0.6$	$u = 0.7$	$u = 0.8$	$u = 0.9$	$u = 1.0$
1	.79697	.75652	.69758	.61571	.50717
2	.44103	.33753	.21334	.07268	−.07467
3	.17349	.05584	−.06712	−.18088	−.26516
4	.01512	−.08442	−.17102	−.22623	−.23096
5	−.05858	−.12880	−.17365	−.17745	−.13069
6	−.08277	−.12515	−.13706	−.10917	−.04351
7	−.08263	−.10328	−.09387	−.05217	.01053
8	−.07238	−.07784	−.05696	−.01349	.03573
9	−.05914	−.05494	−.02954	.00899	.04194
10	−.04621	−.03648	−.01097	.01974	.03785
12	−.02564	−.01254	.00704	.02185	.02025
14	−.01264	−.00112	.01088	.01463	.00571
16	−.00526	.00325	.00923	.00728	−.00172
18	−.00142	.00421	.00625	.00231	−.00401
20	.00037	.00374	.00359	−.00032	−.00369

TABLE 3

Values of Modified Mathieu Function $Ce_2(u, q)$

q	$u = 0.1$	$u = 0.2$	$u = 0.3$	$u = 0.4$	$u = 0.5$
1	1.0988	1.1373	1.2008	1.2883	1.3974
2	1.0549	1.0724	1.0988	1.1297	1.1585

TABLE 3 (*Continued*)

3	0.96379	0.96299	0.95773	0.94205	0.90754
4	0.85071	0.83405	0.80203	0.74858	0.66614
5	0.72581	0.69617	0.64300	0.56150	0.44693
6	0.59925	0.56021	0.49256	0.39375	0.26316
7	0.48147	0.43689	0.36159	0.25581	0.12340
8	0.37974	0.33303	0.25593	0.15146	0.02752
9	0.29657	0.25032	0.17560	0.07791	−0.03165
10	0.23086	0.18674	0.11699	0.02902	−0.06385
12	0.14063	0.10314	0.04636	−0.01995	−0.08086
14	0.08716	0.05686	0.01298	−0.03411	−0.07028
16	0.05511	0.03123	−0.00175	−0.03393	−0.05319
18	0.03549	0.01694	−0.00746	−0.02879	−0.03736
20	0.02324	0.00892	−0.00894	−0.02268	−0.02491
q	$u = 0.6$	$u = 0.7$	$u = 0.8$	$u = 0.9$	$u = 1.0$
1	1.5243	1.6624	1.8011	1.9253	2.0134
2	1.1753	1.1676	1.1197	1.0143	0.83594
3	0.84378	0.73947	0.58490	0.37624	0.12180
4	0.54718	0.38688	0.18749	−0.03624	−0.25175
5	0.29722	0.11677	−0.07877	−0.25853	−0.37664
6	0.10531	−0.06602	−0.22436	−0.33044	−0.34055
7	−0.02480	−0.16773	−0.27310	−0.30268	−0.22806
8	−0.10013	−0.20621	−0.25818	−0.22648	−0.10465
9	−0.13427	−0.20357	−0.21119	−0.14039	−0.00653
10	−0.14150	−0.17901	−0.15482	−0.06588	0.05530
12	−0.11743	−0.11198	−0.05854	0.02440	0.08984
14	−0.08057	−0.05594	−0.00279	0.05087	0.06479
16	−0.04921	−0.02021	0.02043	0.04520	0.02999
18	−0.02702	−0.00098	0.02511	0.02973	0.00504
20	−0.01284	0.00753	0.02147	0.01512	−0.00745

TABLE 4

Values of Modified Mathieu Function $Ce_3(u, q)$

q	$u = 0.1$	$u = 0.2$	$u = 0.3$	$u = 0.4$	$u = 0.5$
1	1.1052	1.2213	1.4222	1.7189	2.1258
2	1.1586	1.2504	1.4052	1.6250	1.9097
3	1.1812	1.2486	1.3585	1.5061	1.6816
4	1.1654	1.2097	1.2780	1.3609	1.4428
5	1.1197	1.1428	1.1740	1.2010	1.2055
6	1.0538	1.0580	1.0563	1.0356	0.97750
7	0.97392	0.96113	0.93084	0.87011	0.76270
8	0.88402	0.85635	0.80166	0.70851	0.56459
9	0.78762	0.74742	0.67297	0.55533	0.38747
10	0.68880	0.63876	0.54956	0.41562	0.23594
12	0.50147	0.44023	0.33678	0.19314	0.02033
14	0.34898	0.28634	0.18502	0.05393	−0.08713
16	0.23815	0.18014	0.09012	−0.01830	−0.12097
18	0.16205	0.11131	0.03578	−0.04848	−0.11669
20	0.11083	0.06792	0.00673	−0.05603	−0.09738
q	$u = 0.6$	$u = 0.7$	$u = 0.8$	$u = 0.9$	$u = 1.0$
1	2.6602	3.3395	4.1775	5.1777	6.3233
2	2.2548	2.6466	3.0569	3.4362	3.7080
3	1.8671	2.0337	2.1383	2.1234	1.9238
4	1.4998	1.4989	1.3999	1.1630	0.76444

TABLE 4 (*Continued*)

5	1.1629	1.0441	0.82247	0.48677	0.06084
6	0.86034	0.66443	0.38210	0.03295	−0.32004
7	0.59306	0.35396	0.05843	−0.24680	−0.47537
8	0.36249	0.10894	−0.16367	−0.38955	−0.47787
9	0.17129	−0.07282	−0.29711	−0.42722	−0.38634
10	0.02180	−0.19459	−0.35593	−0.39057	−0.25026
12	−0.15498	−0.28704	−0.31889	−0.21098	0.01050
14	−0.20395	−0.25165	−0.19309	−0.03416	0.14269
16	−0.18368	−0.17205	−0.07499	0.06593	0.15185
18	−0.13945	−0.09636	0.00146	0.09533	0.10267
20	−0.09483	−0.04108	0.03835	0.08428	0.04596

TABLE 5

Values of Modified Mathieu Function $Ce_4(u, q)$

q	$u = 0.1$	$u = 0.2$	$u = 0.3$	$u = 0.4$	$u = 0.5$
1	1.1086	1.3389	1.7569	2.4173	3.4036
2	1.1402	1.3447	1.7094	2.2701	3.0767
3	1.1750	1.3539	1.6668	2.1337	2.7777
4	1.2090	1.3622	1.6246	2.0030	2.4995
5	1.2357	1.3633	1.5764	1.8712	2.2341
6	1.2485	1.3508	1.5161	1.7329	1.9764
7	1.2436	1.3212	1.4415	1.5868	1.7257
8	1.2209	1.2753	1.3539	1.4351	1.4843
9	1.1830	1.2159	1.2564	1.2812	1.2549
10	1.1330	1.1460	1.1523	1.1277	1.0391
12	1.0056	0.98400	0.93253	0.82970	0.65183
14	0.85343	0.80451	0.71006	0.55534	0.33119
16	0.69134	0.62346	0.50214	0.32192	0.08994
18	0.53665	0.45884	0.32677	0.14485	−0.06528
20	0.40354	0.32374	0.19407	0.02753	−0.14399
q	$u = 0.6$	$u = 0.7$	$u = 0.8$	$u = 0.9$	$u = 1.0$
1	4.8340	6.8683	9.7122	13.6152	18.8542
2	4.1905	5.6763	7.5866	9.9321	12.6332
3	3.6170	4.6510	5.8388	7.0688	8.1249
4	3.1011	3.7660	4.4043	4.8617	4.9166
5	2.6316	2.9989	3.2303	3.1799	2.6894
6	2.2011	2.3346	2.2773	1.9215	1.1991
7	1.8078	1.7636	1.5147	1.0045	0.25483
8	1.4518	1.2788	0.91563	0.35942	−0.29441
9	1.1330	0.87224	0.45490	−0.07303	−0.56620
10	0.84996	0.53553	0.10988	−0.34146	−0.64899
12	0.38468	0.04324	−0.30499	−0.53216	−0.49268
14	0.04888	−0.24184	−0.44407	−0.44014	−0.17824
16	−0.15973	−0.35575	−0.40144	−0.23144	0.09096
18	−0.25421	−0.34593	−0.26929	−0.02886	0.22516
20	−0.26510	−0.26980	−0.12543	0.10202	0.23050

TABLE 6

Values of Modified Mathieu Function $Ce_5(u, q)$

q	$u = 0.1$	$u = 0.2$	$u = 0.3$	$u = 0.4$	$u = 0.5$
1	1.1412	1.5282	2.2710	3.5374	5.6077
2	1.1563	1.5151	2.1935	3.3245	5.1202
3	1.1731	1.5042	2.1202	3.1239	4.6683

TABLE 6 (*Continued*)

4	1.1921	1.4957	2.0516	2.9353	4.2504
5	1.2132	1.4897	1.9873	2.7581	3.8641
6	1.2359	1.4855	1.9267	2.5910	3.5062
7	1.2590	1.4818	1.8680	2.4318	3.1731
8	1.2802	1.4763	1.8091	2.2778	2.8609
9	1.2970	1.4665	1.7472	2.1263	2.5659
10	1.3064	1.4497	1.6800	1.9752	2.2856
12	1.2955	1.3884	1.5250	1.6710	1.7661
14	1.2437	1.2906	1.3455	1.3692	1.3036
16	1.1582	1.1651	1.1512	1.0788	0.90042
18	1.0480	1.0205	0.95063	0.80629	0.55675
20	0.92029	0.86472	0.75164	0.55842	0.27427
q	$u = 0.6$	$u = 0.7$	$u = 0.8$	$u = 0.9$	$u = 1.0$
1	8.9280	14.1878	22.4276	35.1766	54.6063
2	7.8948	12.0901	18.2886	27.1920	39.5195
3	6.9605	10.2507	14.7938	20.7544	28.0285
4	6.1173	8.6426	11.8561	15.6014	19.3790
5	5.3573	7.2398	9.3977	11.5090	12.9574
6	4.6719	6.0181	7.3491	8.2873	8.2676
7	4.0527	4.9552	5.6499	5.7767	4.9121
8	3.4918	4.0311	4.2479	3.8446	2.5754
9	2.9824	3.2293	3.0994	2.3820	1.0082
10	2.5195	2.5361	2.1678	1.2989	0.01418
12	1.7215	1.4339	0.83522	−0.01644	−0.84049
14	1.0801	0.65007	0.04279	−0.56769	−0.86016
16	0.57767	0.11871	−0.37198	−0.67437	−0.54741
18	0.19727	−0.21384	−0.52797	−0.54412	−0.17196
20	−0.07276	−0.38751	−0.51169	−0.31371	0.12777

TABLE 7

Values of Modified Mathieu Function $Se_1(u, q)$

q	$u = 0.1$	$u = 0.2$	$u = 0.3$	$u = 0.4$	$u = 0.5$
1	.06840	.13532	.19914	.25799	.30961
2	.04709	.09159	.13086	.16208	.18239
3	.03295	.06296	.08715	.10274	.10727
4	.02347	.04404	.05895	.06582	.06293
5	.01700	.03131	.04047	.04256	.03661
6	.01251	.02260	.02815	.02771	.02090
7	.00934	.01654	.01981	.01811	.01152
8	.00705	.01224	.01408	.01184	.00593
9	.00538	.00916	.01009	.00772	.00264
10	.00415	.00691	.00728	.00499	.00075
12	.00253	.00404	.00385	.00197	−.00083
14	.00159	.00242	.00207	.00065	−.00111
16	.00102	.00149	.00111	.00009	−.00098
18	.00067	.00093	.00060	−.00012	−.00075
20	.00044	.00059	.00032	−.00018	−.00053
q	$u = 0.6$	$u = 0.7$	$u = 0.8$	$u = 0.9$	$u = 1.0$
1	.35130	.37984	.39160	.38277	.34981
2	.18892	.17923	.15176	.10677	.04736
3	.09899	.07736	.04382	.00256	−.03898
4	.04974	.02742	−.00056	−.02823	−.04767
5	.02305	.00415	−.01572	−.03061	−.03443

TABLE 7 (*Continued*)

6	.00883	−.00566	−.01822	−.02392	−.01929
7	.00153	−.00889	−.01587	−.01590	−.00797
8	−.00197	−.00906	−.01215	−.00920	−.00104
9	−.00341	−.00795	−.00856	−.00440	.00249
10	−.00377	−.00644	−.00559	−.00130	.00381
12	−.00317	−.00366	−.00177	.00138	.00329
14	−.00219	−.00178	−.00004	.00169	.00172
16	−.00136	−.00070	.00054	.00123	.00053
18	−.00079	−.00015	.00061	.00069	−.00007
20	−.00042	.00008	.00049	.00030	−.00028

TABLE 8

Values of Modified Mathieu Function $Se_2(u, q)$

q	$u = 0.1$	$u = 0.2$	$u = 0.3$	$u = 0.4$	$u = 0.5$
1	0.16828	0.33969	0.51700	0.70220	0.89595
2	0.13834	0.27606	0.41188	0.54311	0.66499
3	0.11228	0.22132	0.32313	0.41239	0.48190
4	0.09035	0.17576	0.25071	0.30867	0.34203
5	0.07234	0.13880	0.19312	0.22858	0.23834
6	0.05782	0.10933	0.14814	0.16794	0.16323
7	0.04622	0.08610	0.11343	0.12265	0.10981
8	0.03702	0.06788	0.08682	0.08914	0.07235
9	0.02973	0.05364	0.06650	0.06449	0.04641
10	0.02396	0.04251	0.05100	0.04642	0.02867
12	0.01573	0.02696	0.03013	0.02361	0.00874
14	0.01048	0.01733	0.01791	0.01154	0.00030
16	0.00709	0.01128	0.01068	0.00523	−0.00276
18	0.00486	0.00743	0.00638	0.00201	−0.00345
20	0.00337	0.00495	0.00380	0.00043	−0.00317
q	$u = 0.6$	$u = 0.7$	$u = 0.8$	$u = 0.9$	$u = 1.0$
1	1.0969	1.3008	1.4992	1.6785	1.8187
2	0.76997	0.84717	0.88237	0.85901	0.76084
3	0.52245	0.52335	0.47414	0.36785	0.20632
4	0.34263	0.30332	0.22081	0.09992	−0.04168
5	0.21658	0.16068	0.07444	−0.02837	−0.12215
6	0.13090	0.07253	−0.00276	−0.07653	−0.12279
7	0.07428	0.02093	−0.03795	−0.08311	−0.09329
8	0.03790	−0.00711	−0.04926	−0.07112	−0.05861
9	0.01528	−0.02059	−0.04809	−0.05321	−0.02935
10	0.00180	−0.02546	−0.04128	−0.03567	−0.00851
12	−0.00948	−0.02312	−0.02421	−0.01017	0.01098
14	−0.01095	−0.01606	−0.01094	0.00207	0.01293
16	−0.00912	−0.00960	−0.00310	0.00583	0.00876
18	−0.00664	−0.00501	0.00069	0.00557	0.00415
20	−0.00446	−0.00214	0.00207	0.00396	0.00098

TABLE 9

Values of Modified Mathieu Function $Se_3(u, q)$

q	$u = 0.1$	$u = 0.2$	$u = 0.3$	$u = 0.4$	$u = 0.5$
1	0.28177	0.58335	0.92520	1.3289	1.8174
2	0.25585	0.52460	0.81849	1.1481	1.5208
3	0.22907	0.46514	0.71366	0.97678	1.2509
4	0.20280	0.40767	0.61469	0.81981	1.0124

TABLE 9 (*Continued*)

5	0.17782	0.35376	0.52376	0.67953	0.80655
6	0.15465	0.30434	0.44202	0.55677	0.63263
7	0.13358	0.25990	0.36990	0.45135	0.48858
8	0.11473	0.22060	0.30732	0.36236	0.37149
9	0.09810	0.18633	0.25380	0.28836	0.27798
10	0.08362	0.15680	0.20856	0.22765	0.20449
12	0.06038	0.11023	0.13930	0.13881	0.10419
14	0.04348	0.07713	0.09212	0.08232	0.04723
16	0.03134	0.05394	0.06054	0.04734	0.01669
18	0.02267	0.03780	0.03961	0.02616	0.00146
20	0.01648	0.02658	0.02582	0.01359	−0.00527
q	$u = 0.6$	$u = 0.7$	$u = 0.8$	$u = 0.9$	$u = 1.0$
1	2.4142	3.1416	4.0172	5.0475	6.2179
2	1.9378	2.3902	2.8537	3.2814	3.5986
3	1.5235	1.7698	1.9495	2.0066	1.8751
4	1.1728	1.2712	1.2683	1.1222	0.80428
5	0.88305	0.88107	0.77178	0.53732	0.18998
6	0.64900	0.58420	0.42370	0.17408	−0.12034
7	0.46430	0.36502	0.19076	−0.03205	−0.24102
8	0.32187	0.20848	0.04385	−0.13250	−0.25372
9	0.21456	0.10082	−0.04135	−0.16629	−0.21335
10	0.13560	0.03006	−0.08430	−0.16155	−0.15413
12	0.03941	−0.03848	−0.09867	−0.10630	−0.04642
14	−0.00455	−0.05363	−0.07407	−0.04790	0.01347
16	−0.02077	−0.04703	−0.04423	−0.00946	0.03233
18	−0.02360	−0.03432	−0.02094	0.00930	0.02931
20	−0.02082	−0.02216	−0.00612	0.01502	0.01890

TABLE 10

Values of Modified Mathieu Function $Se_4(u, q)$

q	$u = 0.1$	$u = 0.2$	$u = 0.3$	$u = 0.4$	$u = 0.5$
1	0.39463	0.84503	1.4138	2.1776	3.2347
2	0.37627	0.79817	1.3148	1.9819	2.8633
3	0.35586	0.74793	1.2133	1.7897	2.5130
4	0.33385	0.69530	1.1107	1.6029	2.1858
5	0.31081	0.64145	1.0090	1.4239	1.8836
6	0.28730	0.58752	0.90979	1.2547	1.6076
7	0.26380	0.53448	0.81449	1.0967	1.3585
8	0.24073	0.48314	0.72419	0.95107	1.1363
9	0.21842	0.43413	0.63969	0.81833	0.94041
10	0.19712	0.38791	0.56153	0.69870	0.76968
12	0.15832	0.30511	0.42530	0.49800	0.49758
14	0.12520	0.23597	0.31565	0.34487	0.30507
16	0.09787	0.18015	0.23039	0.23236	0.17521
18	0.07590	0.13626	0.16591	0.15236	0.09165
20	0.05858	0.10243	0.11819	0.09707	0.04051
q	$u = 0.6$	$u = 0.7$	$u = 0.8$	$u = 0.9$	$u = 1.0$
1	4.7138	6.7818	9.6490	13.5682	18.8187
2	4.0298	5.5534	7.4914	9.8579	12.5758
3	3.4110	4.4900	5.7143	6.9759	8.0609
4	2.8560	3.5782	4.2677	4.7731	4.8735
5	2.3634	2.8050	3.1063	3.1223	2.6921
6	1.9310	2.1575	2.1891	1.9148	1.2588

TABLE 10 (*Continued*)

7	1.5561	1.6228	1.4782	1.0577	0.36942
8	1.2350	1.1880	0.93945	0.47267	−0.13593
9	0.96346	0.84048	0.54207	0.09435	−0.38047
10	0.73712	0.56804	0.25870	−0.13131	−0.45716
12	0.40088	0.20320	−0.05883	−0.29067	−0.35966
14	0.18811	0.01168	−0.16636	−0.25361	−0.17308
16	0.06334	−0.07176	−0.16852	−0.15892	−0.02863
18	−0.00295	−0.09442	−0.12925	−0.07123	0.04656
20	−0.03308	−0.08730	−0.08258	−0.01164	0.06763

TABLE 11

Values of Modified Mathieu Function $Se_5(u, q)$

q	$u = 0.1$	$u = 0.2$	$u = 0.3$	$u = 0.4$	$u = 0.5$
1	0.50806	1.1350	2.0261	3.3842	5.5114
2	0.49404	1.0935	1.9233	3.1498	5.0058
3	0.47893	1.0503	1.8205	2.9229	4.5311
4	0.46266	1.0055	1.7175	2.7032	4.0855
5	0.44522	0.95900	1.6143	2.4904	3.6675
6	0.42665	0.91094	1.5113	2.2847	3.2764
7	0.40710	0.86162	1.4089	2.0865	2.9116
8	0.38675	0.81145	1.3076	1.8965	2.5728
9	0.36580	0.76086	1.2082	1.7154	2.2599
10	0.34451	0.71035	1.1113	1.5437	1.9724
12	0.30177	0.61133	0.92761	1.2307	1.4718
14	0.26020	0.51760	0.76060	0.96037	1.0654
16	0.22112	0.43160	0.61301	0.73302	0.74519
18	0.18547	0.35491	0.48605	0.54707	0.50073
20	0.15383	0.28828	0.37963	0.39910	0.32040
q	$u = 0.6$	$u = 0.7$	$u = 0.8$	$u = 0.9$	$u = 1.0$
1	8.8670	14.1488	22.4024	35.1601	54.5952
2	7.8187	12.0384	18.2527	27.1664	39.5007
3	6.8646	10.1820	14.7432	20.7166	28.0002
4	5.9972	8.5532	11.7889	15.5516	19.3442
5	5.2102	7.1291	9.3159	11.4525	12.9247
6	4.4984	5.8894	7.2595	8.2343	8.2493
7	3.8571	4.8161	5.5633	5.7400	4.9202
8	3.2821	3.8927	4.1773	3.8368	2.6181
9	2.7695	3.1041	3.0573	2.4121	1.0872
10	2.3154	2.4362	2.1638	1.3709	0.12471
12	1.5673	1.4110	0.92094	0.13225	−0.69719
14	1.0064	0.72162	0.21470	−0.37207	−0.73439
16	0.60145	0.28547	−0.13573	−0.47604	−0.48918
18	0.32205	0.03208	−0.26611	−0.39470	−0.21558
20	0.13950	−0.09651	−0.27382	−0.25559	−0.01570

TABLE 12

Values of Modified Mathieu Function $Se_6(u, q)$

q	$u = 0.1$	$u = 0.2$	$u = 0.3$	$u = 0.4$	$u = 0.5$
1	0.62524	1.4688	2.8234	5.1539	9.2485
2	0.61333	1.4278	2.7067	4.8536	8.5252
3	0.60091	1.3863	2.5917	4.5649	7.8454
4	0.58792	1.3442	2.4785	4.2872	7.2067
5	0.57431	1.3015	2.3667	4.0198	6.6067

TABLE 12 (*Continued*)

6	0.56004	1.2581	2.2562	3.7624	6.0431
7	0.54506	1.2138	2.1469	3.5143	5.5139
8	0.52935	1.1687	2.0387	3.2753	5.0174
9	0.51290	1.1227	1.9317	3.0452	4.5517
10	0.49573	1.0760	1.8258	2.8237	4.1161
12	0.45945	0.98057	1.6184	2.4068	3.3294
14	0.42114	0.88388	1.4184	2.0256	2.6494
16	0.38169	0.78772	1.2283	1.6812	2.0693
18	0.34208	0.69405	1.0507	1.3748	1.5822
20	0.30326	0.60465	0.88754	1.1066	1.1806
q	$u = 0.6$	$u = 0.7$	$u = 0.8$	$u = 0.9$	$u = 1.0$
1	16.4676	29.1545	51.3078	89.6491	155.255
2	14.7990	25.4199	43.1373	72.1129	118.287
3	13.2690	22.0897	36.0874	57.5680	89.0684
4	11.8672	19.1253	30.0221	45.5612	66.1482
5	10.5841	16.4914	24.8206	35.7016	48.3227
6	9.4108	14.1560	20.3757	27.6527	34.5956
7	8.3393	12.0899	16.5923	21.1260	24.1462
8	7.3621	10.2671	13.3865	15.8741	16.3000
9	6.4725	8.6636	10.6839	11.6855	10.5056
10	5.6645	7.2582	8.4189	8.3797	6.3134
12	4.2713	4.9650	4.9760	3.8237	1.3490
14	3.1428	3.2514	2.6685	1.2321	−0.73178
16	2.2438	2.0028	1.1913	−0.09221	−1.28226
18	1.5421	1.1221	0.30627	−0.63917	−1.12283
20	1.0078	0.52675	−0.17078	−0.74531	−0.72343